Fermionic Schwinger-Keldysh propagators from AdS/CFT

Fermionic Schwinger-Keldysh propagators from AdS/CFT

G.C. Giecold

Institut de Physique Théorique,
CEA Saclay, CNRS (URA 2306), F-91191 Gif-sur-Yvette, France

E-mail: gregory.giecold@cea.fr

ABSTRACT: The Herzog and Son prescription [1] for computing real-time Green functions for finite temperature gauge theories from their gravity dual is generalized to fermions. These notes explain how such an extension involves properties of spinors in a curved, complexified space-time.

Contents

1 Introduction

The gauge-string duality relates some classes of field theories to dual string theories in specified background space-times [2]. While string theory in a curved background does not generally lends itself to tractable calculations and even to our understanding, its low-energy supergravity limit is far more compliant. In the case of asymptotically AdS spaces, the dual field theories are in the large 't Hooft coupling limit. The AdS/CFT correspondence thus provides a framework for understanding strongly-coupled gauge theories. Recent work has been devoted to computing transport coefficients [3–8] and gaining insight into dynamic and nonequilibrium settings [9, 10] from the correspondence. Most of them required a real-time formulation of finite temperature field theory. The way real-time correlators can be derived in AdS/CFT is hinted by the following analogy. There is a doubling of the degrees of freedom in the Schwinger-Keldysh real-time prescription (reviewed in section 2 of this paper). On the other hand, the Penrose diagrams of asymptotically AdS spacetimes with a black hole[1] exhibit two boundaries (cf. figure 1. below), on which the dual gauge theory fields live. This conjecture was proved by Herzog and Son [1]. They showed how the 2×2 matrix of two-point correlation functions for a scalar field and its doubler partner field is reproduced from the AdS dual supergravity action. Their work also made it clear that the thermal nature of black hole physics gives rise to the thermal nature of its dual field theory. In these notes, we would like to extend their work and find out how black hole physics gives rise to real-time correlators of fermionic operators in a dual finite-temperature field theory. While deriving real-time propagators of vector field operators from AdS/CFT is an obvious extension of the scalar field case expounded in [1], the case of fermions proves less straightforward. The analysis presented below relies on a treatment of spinor fields in curved space-times and on their transformation laws under global symmetry transformations. We begin by reviewing in the next section the Schwinger-Keldysh formalism for

---

[1]Whose temperature is that of the dual gauge theory, according to the AdS/CFT correspondence.
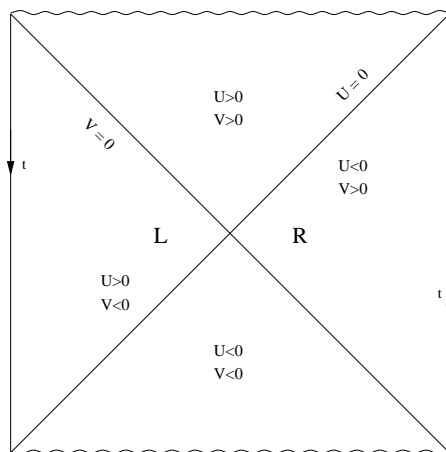
Figure 1. Kruskal diagram for AdS-Schwarzschild black hole.

real-time finite temperature field theory. This section collects results usually dispersed among the literature, as this formalism is usually exposed for a scalar field. We present the 2×2 matrix of two-point correlation functions for a fermionic operator and its link with the retarded and advanced propagators. Section 3 reviews how the retarded Green function for a fermionic operator at strong coupling can be computed from the dual supergravity spinor classical action in AdS/CFT. Section 4 is devoted to a short review of spinors in curved and complexified space-times. These are important for the analysis of section 5 where we review the relationship between positive– and negative-energy modes of a wave equation and analyticity conditions in the complex Kruskal planes of the underlying background. These conditions lead to the real-time propagators for fermionic operators from the dual boundary action in the gauge-gravity duality.

Recently, there has been a sustained interest in fermions from theories with gravity duals. In [11] the two-point function for a fermionic operator in a non-relativistic conformal field theory is computed. The gravity dual corresponds to fermions propagating in a background with the Schrödinger isometry. The authors of [12] argue that the gauge-gravity correspondence proves a useful tool for exploring fermionic quantum phase transitions. The retarded fermion Green function is found from an analysis of the solutions to the Dirac equation and its quasi-normal modes in an AdS Reissner-Nordström black hole. Real-time correlators for non-relativistic holography have been considered recently in [13], where the construction of [14, 15] is involved. For an explanation of how their construction generalizes the one due to [1] used here to the case of distinct sources on the R and L boundaries of a Penrose diagram, see [16]. It would be interesting to apply the approach contained in the present notes to more general geometries, possibly duals to non-relativistic conformal field theories [17–20]. This paper offers to explain how Schwinger-Keldysh n-point functions can be computed from string theory, thus emphasizing the latter as a relevant approach to tackle some models or phases of condensed matter physics.
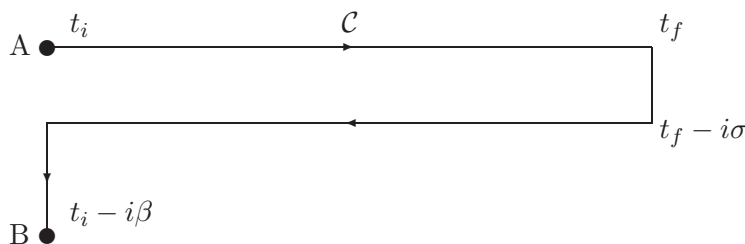
Figure 2. The Schwinger-Keldysh contour.

2 Review of Schwinger-Keldysh formalism for fermions

The Schwinger-Keldysh prescription allows for a study of real-time Green functions by introducing a contour $\mathcal{C}$ in the complex time plane [21, 22], as illustrated on figure 2. Fields live on this time contour. The forward and return contour of the path are labelled by indices i_1 and i_2 respectively. The idea is that the quantum dynamics does the doubling of the degrees of freedom required for describing non-equilibrium states. The starting point I at time t_i and the ending point B at $t_i - i\beta$ are identified and fermionic fields are such that $\Upsilon_I = -\Upsilon_B$. In the remainder of this paper, the conventions for the propagators are those of [23].

The action splits into contributions from the four parts of the contour:

$$
\begin{aligned}
S &= \int_{\mathcal{C}} dt_{\mathcal{C}} L(t_{\mathcal{C}}), \\
&= \int_{t_i}^{t_f} dt L(t) - i \int_0^{\sigma} d\tau L(t_f - i\tau) - \int_{t_i}^{t_f} dt L(t - i\sigma) - i \int_{\sigma}^{\beta} d\tau L(t_i - i\tau), \quad (2.1)
\end{aligned}
$$

where

$$
L(t) = \int d^{d-1}\vec{x} \mathcal{L} \left[\Upsilon(t, \mathbf{x}), \bar{\Upsilon}(t, \mathbf{x}) \right]. \quad (2.2)
$$

The generating functional is defined as

$$
Z = \int \mathcal{D}\Upsilon \mathcal{D}\bar{\Upsilon} \exp\left(iS + i \int_{t_i}^{t_f} d^d x \bar{\eta}_1 \Upsilon_1 + i \int_{t_i}^{t_f} d^d x \bar{\Upsilon}_1 \eta_1 - i \int_{t_i}^{t_f} d^d x \bar{\eta}_2 \Upsilon_2 - i \int_{t_i}^{t_f} d^d x \bar{\Upsilon}_2 \eta_2 \right), \quad (2.3)
$$

where the sources $\eta_{1,2}$ and the fields are such that

$$
\begin{cases}
\eta_1(t, \mathbf{x}) = \eta(t, \mathbf{x}), & \Upsilon_1(t, \mathbf{x}) = \Upsilon(t, \mathbf{x}), \\
\eta_2(t, \mathbf{x}) = \eta(t - i\sigma, \mathbf{x}), & \Upsilon_2(t, \mathbf{x}) = \Upsilon(t - i\sigma, \mathbf{x}).
\end{cases} \quad (2.4)
$$

The same relations hold for their conjugates. The contour-ordered Green functions are mapped into a matrix whose components are indexed by the position on the contour:

$$
iG(j, k) = \frac{1}{i^2} \frac{\delta^2 \ln Z [\eta_{1,2}, \bar{\eta}_{1,2}]}{\delta \eta_j \delta \eta_k^{\dagger}} = i \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \quad (2.5)
$$

The time in the components of this matrix of Green function is standard time and in the operator formalism

$$
\begin{cases}
G_{11}(t,\mathbf{x}) = -i\langle T\Upsilon(t,\mathbf{x})\Upsilon^\dagger(0)\rangle, & G_{12}(t,\mathbf{x}) = +i\langle\Upsilon^\dagger(0)\Upsilon(t,\mathbf{x})\rangle, \\
G_{21}(t,\mathbf{x}) = -i\langle\Upsilon(t,\mathbf{x})\Upsilon^\dagger(0)\rangle, & G_{22}(t,\mathbf{x}) = -i\langle\hat{T}\Upsilon(t,\mathbf{x})\Upsilon^\dagger(0)\rangle.
\end{cases}
\tag{2.6}
$$

Note the sign reversal in G_{12} as compared to the case where the fields are bosonic. T and $\hat{T}$ denote the time-ordering and anti-time-ordering operators. The fields are taken in the Heisenberg picture. Those Schwinger-Keldysh correlators are related to the retarded and advanced Green functions through

$$
\begin{cases}
G_R(x-y) = -i\theta(x^0 - y^0)\langle\{\Upsilon(x), \Upsilon^\dagger(y)\}\rangle, \\
G_A(x-y) = +i\theta(y^0 - x^0)\langle\{\Upsilon(x), \Upsilon^\dagger(y)\}\rangle.
\end{cases}
\tag{2.7}
$$

The relation

$$
G_R(x-y) = G_A^*(y-x)
\tag{2.8}
$$

is valid, irrespective of the fields obeying a Bose-Einstein or Fermi-Dirac statistics. Using (2.6), (2.7) and the completeness relation for a complete set of state results in

$$
\begin{cases}
G_{11}(k) = -\mathrm{Re}G_R(k) + i\tanh\left(\frac{k^0}{2T}\right)\mathrm{Im}G_R(k), \\
G_{12}(k) = -\frac{2ie^{\sigma k^0}}{1+e^{\beta k^0}}\mathrm{Im}G_R(k), \\
G_{21}(k) = \frac{2ie^{(\beta-\sigma)k^0}}{1+e^{\beta k^0}}\mathrm{Im}G_R(k), \\
G_{22}(k) = \mathrm{Re}G_R(k) + i\tanh\left(\frac{k^0}{2T}\right)\mathrm{Im}G_R(k).
\end{cases}
\tag{2.9}
$$

When $\sigma = \frac{\beta}{2}$ — a value which will naturally appear in the following — $G_{12}(k) = -G_{21}(k)$. $\sigma = 0$ yields $G_{21}(k) = G^>(k)$ and $G_{12}(k) = G^<(k)$. Since $G_{21}(k) - G_{12}(k)\mid_{\sigma=0} = 2i\mathrm{Im}G_R(k)$, the relationship

$$
G_R(k) - G_A(k) = G^>(k) - G^<(k)
\tag{2.10}
$$

holds as required whatever the quantum statistics of the fields under consideration. The main purpose of these notes is to show how the above relations (2.9) are derived in AdS/CFT. Herzog and Son [1] obtain analogous relations for a scalar field. As it turns out, extending their result to correlators of fermionic operators is not entirely straightforward.

3 Review of fermionic retarded correlators in AdS/CFT

This section reviews fermions in AdS/CFT [24, 25]. The prescription for computing retarded fermionic Green functions draws on the approach of Iqbal and Liu [26, 27], where conjugate momenta for supergravity fields are defined with respect to a r-foliation. This is suggestive of some sort of stochastic quantization interpretation of the AdS/CFT correspondence.

Consider a boundary fermionic operator $\mathcal{O}$ whose gravity dual is a spinor field Ψ. The bulk background space-time metric

$$ds^2 = g_{rr}dr^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{3.1}$$

is subjected to the asymptotically-AdS conditions

$$g_{tt}, g_{ii} \sim r^2, \quad g_{rr} \sim 1/r^2, \quad r \to \infty. \tag{3.2}$$

The d-dimensional boundary is a $r \to \infty$. The AdS/CFT prescription for computing n-point functions of a quantum field theory from a classical supergravity action goes as

$$\left\langle \exp\left[\int d^d x \left(\bar{\chi}_0 \mathcal{O} + \bar{\mathcal{O}} \chi_0\right)\right] \right\rangle_{QFT} = e^{iS_{SUGRA}[\chi_0, \bar{\chi}_0]}, \tag{3.3}$$

where $\chi_0 = \lim_{r \to \infty} r^{d-\Delta} \Psi$ and Δ is the scaling dimension of $\mathcal{O}$, related to the mass m of the bulk spinor. Imposing such a Dirichlet boundary condition on spinors requires care, especially given that this condition must relate a spinor in the bulk in $d+1$ dimensions to one in d space-time dimensions on the boundary. Given that our focus is on Green functions, the quadratic part of the action for ψ meets our purpose. It is given by

$$S = i \int_M d^{d+1}x \sqrt{-g} \left(\bar{\Psi}\Gamma^M D_M \Psi - m\bar{\Psi}\Psi\right) + S_{\partial M}, \tag{3.4}$$

with $\bar{\Psi} = \Psi^\dagger \Gamma^t$. The covariant derivative is specified by the spin connection ω_{abM}:

$$D_M = \partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab}, \tag{3.5}$$

where $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$. Upper-case letters stand for abstract space-time indices, while lower-case ones denote tangent frame indices. The two are linked through a choice of vielbein e_M^a, defined by $G_{MN} = e_M^a e_N^b \eta_{ab}$, with η_{ab} a $d+1$-dimensional Minkowski metric of signature $(-,+,+,\ldots,+)$ such that

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab}. \tag{3.6}$$

The inverse vielbein satisfy $\eta_{ab} = G_{MN}e_a^M e_b^N$. The spin connection components are given by $\omega_{abc} = e_a^M \omega_{bcM}$ and $\omega_{abc} = \frac{1}{2}(\mathfrak{C}_{bca} + \mathfrak{C}_{acb} - \mathfrak{C}_{abc})$, where $[e_a, e_b] = \nabla_{e_a}e_b - \nabla_{e_b}e_a = \mathfrak{C}_{ab}^c e_c$. Alternatively ω_{abM} can be viewed as the components of 1-forms ω^M in Cartan's structure equations [28]. Whenever a specific index appears as a label on a Gamma matrix, it refers to that particular index in the tangent frame. For d even, a convenient choice for the bulk Gamma matrices is

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^r = \gamma^{d+1}, \tag{3.7}$$

γ^μ being the boundary gamma matrices and γ^{d+1} being proportional to their product. When d is odd, it is appropriate to choose

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \quad \Gamma^r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.8}$$

They satisfy the Clifford algebra. So, for general d

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \Gamma_\pm \Psi, \quad \Gamma_\pm = \frac{1}{2} \left(1 \pm \Gamma^r\right), \tag{3.9}$$

with $\Psi_\pm$ being opposite chirality Weyl spinors when d is even, and d-dimensional Dirac spinors for d odd. Whatever the value of d the number of components of the fermionic operator $\mathcal{O}$ is always half that of its dual Ψ. Quite generally [29], appendix B, D-dimensional gamma matrices are constructed by noticing that for D even, increasing D by 2 doubles the size of the Dirac matrices. They can therefore be constructed iteratively, starting in $D = 2$ with

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.10}$$

to the following in $D = 2k + 2$

$$\Gamma^\mu = \gamma^\mu \bigotimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^{D-2} = I \bigotimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^{D-1} = I \bigotimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.11}$$

Here, γ^μ, $\mu = 0, \ldots, D-3$ are $2^k \times 2^k$ gamma matrices and I is the $2^k \times 2^k$ identity matrix. When D is odd, simply add $\Gamma^D = \Gamma$ or $-\Gamma$ to the set of $D-1$ gamma matrices. Note that from our conventions for the metric and anti-commutation relations the 0-component of gamma matrices is anti-hermitian while other matrices are hermitian. In order to solve the equations of motion near the boundary and find the scaling dimension Δ of the fermionic operator $\mathcal{O}$, one refers to the usual Frobenius procedure of trying for solutions of the type $r^{-\rho} \sum_{n=0}^\infty \Psi_n(t, x^i)/r^n$. Ψ_n are boundary spinors. Consider for instance the case of pure AdS, $ds^2 = r^2(-dt^2 + d\mathbf{x}^2) + \frac{dr^2}{r^2}$. Setting $e^\mu = r dx^\mu$, $e^r = \frac{dr}{r}$, the non-vanishing spin coefficients

$$\omega_{tr} = -r dt, \quad \omega_{ir} = r dx^i. \tag{3.12}$$

The Dirac equation

$$\left[\slashed{D} - m\right] \Psi = 0, \tag{3.13}$$

with $\slashed{D} = \Gamma^M D_M$, becomes

$$r \Gamma^r \partial_r \Psi + \frac{i}{r} \Gamma \cdot k \Psi + \frac{d}{2} \Gamma^r \Psi - m \Psi = 0, \tag{3.14}$$

where $\Gamma \cdot k = \gamma^\mu k_\mu$. Then ρ must be set to $\rho = \frac{d}{2} \pm m$ and Ψ_0 is annihilated by $\frac{1}{2}\left(1 \pm \Gamma^r\right)$, respectively. Incidentally, the scaling dimension is therefore found to be $\Delta = \frac{d}{2} + m$. The leading asymptotic behaviour of $\Psi_\pm$ is then

$$\Psi_+(k, r) = \chi_0(k) r^{m - \frac{d}{2}} + \lambda_0(k) r^{-\frac{d}{2} - m - 1}, \Psi_-(k, r) = \psi_0(k) r^{-\frac{d}{2} - m} + \mu_0(k) r^{m - \frac{d}{2} - 1} \tag{3.15}$$

Note that the dominant term (when $m \geq 0$) has been denoted χ_0 on purpose, as a reference to the source for the dual operator $\mathcal{O}$ in (3.3):

$$\lim_{r \to \infty} r^{d-\Delta} \Psi_+ = \chi_0. \tag{3.16}$$

– 6 –

Inserting this back into their equation of motion yields

$$
\begin{cases}
\psi_0(k) = -\frac{i\gamma \cdot k}{k^2}(1 + 2m)\lambda_0(k); \\
\mu_0(k) = -\frac{i\gamma \cdot k}{1 - 2m}\chi_0(k).
\end{cases}
\tag{3.17}
$$

If one then demands that the solution be regular in the whole of AdS space, it turns out that χ_0 and ψ_0 are not independent. Note that this is not apparent from the analysis presented above where χ_0 and ψ_0 are the boundary values of the fields Ψ_+ and Ψ_-. A general solution to the Dirac equation near the boundary is a superposition of those fields. However the Dirac equation can be solved exactly in a few cases, including pure AdS. In this latter case, suitable solutions are given by

$$
\Psi_+ =
\begin{cases}
r^{-\frac{d+1}{2}} K_{m+\frac{1}{2}}\left(\frac{\sqrt{\mathbf{k}^2 - \omega^2}}{r}\right)\kappa_+, & k^2 > 0, \\
r^{-\frac{d+1}{2}} H^{(1)}_{m+\frac{1}{2}}\left(\frac{\sqrt{\omega^2 - \mathbf{k}^2}}{r}\right)\kappa_+, & \omega > \sqrt{\mathbf{k}^2}, \\
r^{-\frac{d+1}{2}} H^2_{m+\frac{1}{2}}\left(\frac{\sqrt{\omega^2 - \mathbf{k}^2}}{r}\right)\kappa_+, & \omega < -\sqrt{\mathbf{k}^2},
\end{cases}
\tag{3.18}
$$

κ_+ denoting a constant spinor. Regularity as $r \to 0$ then imposes that

$$
\psi_0(k) = -\frac{i\gamma \cdot k}{k^2}\frac{(\frac{k}{2})^{2m}\Gamma(\frac{1}{2} - m)}{\Gamma(\frac{1}{2} + m)}\chi_0(k).
\tag{3.19}
$$

More generally ψ_0 and χ_0 are related by a matrix $\mathcal{S}$:

$$
\psi_0(k) = \mathcal{S}(k)\chi_0(k).
\tag{3.20}
$$

Regularity in the bulk and a given boundary condition χ_0 for Ψ_+ when $m \geq 0$ then uniquely determine a solution Ψ to the classical equations of motion. Similar relations apply for $\bar{\Psi}_+$ and $\bar{\Psi}_-$. Note that relations such as (3.19) or (3.20) are on-shell relations. Other off-shell histories contributing to the variational principle can be constructed as superpositions of independent Ψ_+ and Ψ_-. We now turn to a discussion of the variational principle. The boundary term in (3.4) will be determined from stationarity of the action. That one cannot fix all the components of Ψ and $\bar{\Psi}$ but must rather set conditions on, say, χ_0, $\bar{\chi}_0$, and leave ψ_0, $\bar{\psi}_0$ free to vary,[2] stems from the Dirac equation being first order in derivatives. Varying with respect to Ψ, $\bar{\Psi}$ the Euclidean action

$$
S = -\int_M d^{d+1}x \sqrt{g}\,\bar{\Psi}\left(\frac{1}{2}\left(\overrightarrow{\not{D}} - \overleftarrow{\not{D}}\right) - m\right)\Psi,
\tag{3.21}
$$

one finds a surface term, i.e. $\delta S = C_{\partial M}$+ bulk term, where the bulk term involves radial derivatives and is proportional to the equations of motions, while

$$
\begin{aligned}
C_{\partial M} &= \frac{1}{2}\int_{\partial M} d^d x\left(\delta\bar{\psi}_0\chi_0 + \bar{\chi}_0\delta\psi_0\right)(x), \\
&= \delta\left\{\frac{1}{2}\int_{\partial M} d^d x\left(\bar{\psi}_0\chi_0 + \bar{\chi}_0\psi_0\right)(x)\right\},
\end{aligned}
\tag{3.22}
$$

---

[2]On-shell they become functions of the boundary conditions χ_0, $\bar{\chi}_0$.

given that $\delta\chi_0 = 0$ and $\delta\bar{\chi}_0 = 0$, as explained above. Since $C_{\partial M}$ does not vanish, that the action must be stationary requires that one adds to it a boundary term $S_{\partial M}$ such that $C_{\partial M} = -\delta S_{\partial M}$. Actually, since they are fixed, one may add any function of χ_0 and $\bar{\chi}_0$ to the action without breaking the stationarity condition. However conditions such as locality, absence of derivatives and invariance under the asymptotically AdS symmetries seem to uniquely select a boundary term [25]. Hence, after another Wick-rotation to go back to a Lorentzian signature,

$$S_{\partial M} = -i \int_{\partial M} d^d x \sqrt{g g^{rr}} \, \bar{\Psi}_+ \Psi_-. \tag{3.23}$$

The factor of g^{rr} entering the square root comes from the vielbein. We now review the prescription derived by Iqbal and Son [26, 27] for computing retarded propagators. It amounts to taking Euclidean canonical momenta conjugate to $\Psi_\pm$ with respect to a r-foliation:

$$\Pi_+ = -\sqrt{g g^{rr}} \, \bar{\Psi}_-, \quad \Pi_- = -\sqrt{g g^{rr}} \, \bar{\Psi}_+. \tag{3.24}$$

Then (3.15) and (3.16) result in

$$\begin{aligned} \langle \mathcal{O} \rangle_{\chi_0} &= - \lim_{r \to \infty} r^{\Delta - d} \Pi_+, \\ &= \bar{\psi}_0. \end{aligned} \tag{3.25}$$

From (3.20), i.e. $\psi_0 = \mathcal{S}\chi_0$, and analytic continuation, one obtains retarded correlators in Lorentzian signature:

$$G_R(k) = i\mathcal{S}(k)\gamma^t. \tag{3.26}$$

The γ^t gamma matrix arises since $G_R \sim \langle \mathcal{O}\mathcal{O}^\dagger \rangle$, rather than $\langle \mathcal{O}\bar{\mathcal{O}} \rangle$.

4   Spinors in complexified space-time

In order to generalize the work of Herzog and Son [1] to fermions, it is necessary to consider spinors in curved space-time. Besides, a potential difficulty arises given that [1] relies crucially on analycity of complexified Kruskal coordinates. [30], section 6.9, has a few pages devoted to spinors in complex space-times. [31, 32] provide a complementary treatment of spinors and twistors. A spin space $\aleph$ of complex dimension two comes with each point of the underlying space-time manifold. The members of such a space are negative-chirality, dotted, say, Weyl spinors. Undotted spinors are members of the complex conjugate space $\bar{\aleph}$. The manifold being complexified, $\aleph$ and $\bar{\aleph}$ must be viewed as independent spaces, so that pairs of spinors ξ^α and $\bar{\xi}^{\dot\alpha}$ which previously determined one another under complex conjugation, are replaced by a pair of independent such spinors. A complexified space-time originates from a real underlying space-time by allowing its coordinates to take on complex values and by extending the metric coefficients analytically to the complex domain. Note that this is distinct from a complex space-time where generally no subspace can be singled out as real. Defining a spinor basis or dyad $\{\varsigma^\alpha, \iota^\alpha\}$ for $\aleph$ and $\{\bar{\varsigma}^{\dot\alpha}, \bar{\iota}^{\dot\alpha}\}$ for $\bar{\aleph}$, each comes

bestowed with its own indices-lowering $\epsilon_{\alpha\beta}$ spinor or $\bar{\epsilon}_{\dot\alpha\dot\beta}$ spinor, such that $\epsilon(\varsigma, \imath) = \gamma$ and $\bar{\epsilon}(\bar\varsigma, \bar\imath) = \bar\gamma$. When $\gamma = 1$, the dyad is called a spin-frame. Associated with any spin frame is a null tetrad

$$l = \varsigma\bar\varsigma, \quad m = \varsigma\bar\imath, \quad n = \imath\bar\imath, \quad \bar m = \imath\bar\varsigma, \tag{4.1}$$

which spans the tensor product space $\aleph \bigotimes \bar\aleph$. This illustrates the standard connection between world-tensor indices a as a pair of spinor indices, one dotted and one undotted. From ϵ and $\bar\epsilon$, a symmetric metric on T is built such that

$$g(l, n) = 1, \quad g(m, \bar m) = -1, \tag{4.2}$$

with all other scalar products vanishing, i.e.

$$g_{ab} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}. \tag{4.3}$$

$\aleph \bigotimes \bar\aleph$ endowed with this scalar product has the structure of a tangent space to a complexified manifold. In the Newman-Penrose formalism a basis for the tangent space is a null tetrad consisting of two real vectors and one complex-conjugate pair of vectors. Consider the complexified manifold obtained from a metric of type (3.1) describing a geometry with a horizon. Let U and V label its Kruskal coordinates. In the context of an asymptotically AdS black hole geometry they are introduced below around (5.3). Staying general for now, a basis for the tangent space is given by four null vectors

$$\frac{\partial}{\partial U}, \quad \frac{\partial}{\partial V}, \quad \frac{\partial}{\partial \zeta} = e^{i\phi}\cot\frac{\theta}{2}, \quad \frac{\partial}{\partial\bar\zeta} = e^{-i\phi}\cot\frac{\theta}{2}, \tag{4.4}$$

with $\frac{\partial}{\partial\zeta}$ parameterizing the anti-celestial sphere. It is related to $\frac{\partial}{\partial\zeta}$ by an antipodal map. Let us map l, n, m and $\bar m$ to $\frac{\partial}{\partial U}$, $\frac{\partial}{\partial V}$, $\frac{\partial}{\partial\zeta}$ and $\frac{\partial}{\partial\zeta}$, respectively. The spinors ς and $\imath$ are then associated with the null vectors $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial V}$, respectively.

The combinations $\sqrt{V}\imath$ and $\sqrt{-U}\varsigma$ or $\sqrt{-V}\imath$ and $\sqrt{U}\varsigma$ are parallely transported across the full U and V complex planes. Indeed, parallel transport of $U\frac{\partial}{\partial U}$ along $\frac{\partial}{\partial V}$ stems from

$$\nabla_{\frac{\partial}{\partial V}}\left[U\frac{\partial}{\partial U}\right] = \frac{\partial}{\partial V}[U]\frac{\partial}{\partial U} + U\nabla_{\frac{\partial}{\partial V}}\frac{\partial}{\partial U}$$
$$= U\left(\Gamma^U_{UV}\frac{\partial}{\partial U} + \Gamma^V_{UV}\frac{\partial}{\partial V}\right), \tag{4.5}$$

where $\Gamma^U_{UV} = \frac{1}{2}g^{UV}\left(\partial_U g_{VV} + \partial_V g_{UV} - \partial_V g_{UV}\right) = 0$ and $\Gamma^V_{UV} = 0$. As for the covariant derivative with respect to $U\frac{\partial}{\partial U}$

$$\nabla_{U\frac{\partial}{\partial U}}\left[U\frac{\partial}{\partial U}\right] = \left(\frac{\partial}{\partial U}[U] + U\,\Gamma^U_{UU}\right)U\frac{\partial}{\partial U}, \tag{4.6}$$

it is directed along $U\frac{\partial}{\partial U}$. This corresponds to the weaker definition of a curve $\mathcal{C}(s)$ with tangent vector T being a geodesic if $\nabla_T T = \alpha T$, α an arbitrary function on the curve. This definition agrees with the notion of $\mathcal{C}(s)$ being among the straightest curves in a

Riemannian manifold if the change of T is parallel to T. The modification with respect to the more familiar criterion $\nabla_T T = 0$ for geodesics and parallel transport lies in the length of T being generally not conserved under $\nabla_T T = \alpha T$. However, in the case at hand, the two conditions are directly equivalent since $U\frac{\partial}{\partial U}$ is null. In any case it is always possible to parametrize the curve so that the geodesic condition takes on its customary form, provided $\frac{d^2 s'}{ds^2} = \alpha \frac{ds'}{ds}$ under $s \to s'$. With such a parameter change, $U\frac{\partial}{\partial U}$ is parallel transported along two families of curves spanning the whole Kruskal plane and associated respectively with the vector field $\frac{\partial}{\partial V}$ and the one obtained from $U\frac{\partial}{\partial U}$. The same holds for $V\frac{\partial}{\partial V}$. The attendant statement on the related spinor basis follows.

This choice of spinor basis will appear naturally in section 5. $\sqrt{-U}$ and the likes are pivotal in generating Fermi-Dirac distribution functions. We should also note that even though the AdS setting presented below involves spinors in a bulk geometry of dimension five, the present discussion on spinors in a complexified four-dimensional space-time is of relevance due to the decomposition (3.9) of a general bulk spinor evolving in $d+1$ dimensions into d-dimensional spinors. The latter are used in the following as in, e.g., [24–26], and while in five dimensions an extra fifth basis vector should appear in (4.4), it is irrelevant for the present purpose. $\Psi_\pm$ spinors will be expanded in the basis constructed out of Kruskal coordinates and $\imath, \varsigma$.

5 Real-time correlators from gravity

While the results about to be derived below should be applicable to a broader class of finite-temperature field theories with a gravity dual, we focus on theories arising from non-extremal D3 branes. Boundary values of supergravity fields in the resulting AdS$_5$ background provide the $N \to \infty$, $g_{YM}^2 N \to \infty$ limit of $\mathcal{N} = 4$ SU(N) supersymmetric Yang-Mills correlators. In the near-horizon limit the metric on a stack of non-extremal D3 branes reads[3]

$$ds^2 = (\pi T R r)^2 \left(-f(r)dt^2 + d\mathbf{x}^2 \right) + R^2 \frac{dr^2}{r^2 f(r)}, \tag{5.1}$$

where $f(r) = 1 - \frac{1}{r^4}$. The boundary is a $r \to \infty$ and the horizon at $r = 1$. Here, T is the Hawking temperature of this AdS-Schwarzschild black hole and R is the radius of AdS. The analysis of [1] relies on the behaviour of a scalar field in this background. Near the horizon, $\frac{r}{\pi T} = 1 + \epsilon$, solutions to the wave equation behave as $e^{ik^0 r^*}$ and its conjugate, with $k^0 = \omega$ and r^* being the tortoise coordinate:

$$\frac{dr^*}{dr} = \frac{1}{\pi T} \frac{1}{r^2 f(r)}, \quad r^* = \frac{1}{2\pi T} \left(\arctan(r) + \log \sqrt{\frac{r-1}{r+1}} \right). \tag{5.2}$$

The Kruskal coordinates are defined as

$$\begin{cases} U = -\frac{e^{-2\pi T(t+r^*)}}{2\pi T}, \\ V = \frac{e^{2\pi T(t-r^*)}}{2\pi T}. \end{cases} \tag{5.3}$$

---

[3]Setting $R = 1$ for convenience.

The Penrose diagram of figure 1. is constructed from these coordinates. The retarded and advanced solutions comport themselves as

$$\begin{cases} e^{-i\omega t} f(k,r) \sim e^{-\frac{i\omega}{2\pi T} \ln(V)}, & \text{in-falling,} \\ e^{-i\omega t} f^*(k,r) \sim e^{\frac{i\omega}{2\pi T} \ln(-U)}, & \text{out-going.} \end{cases} \tag{5.4}$$

When we considered solutions to the wave equation, we were working in the R-quadrant, $U < 0$, $V > 0$. However, as explained in [1] if one extends the mode functions to the complex U and V planes, one finds that positive-frequency solutions to the wave equation are analytic in the lower U and V complex planes. A solution is composed of only negative-frequency modes provided it is analytic in the upper U and V planes. With regard to the modes of (5.4) one then requires that the solution be analytic in the lower V plane and the upper U plane. Since in the $r - t$ coordinates one can solve the wave equation independently in the R and L regions of the Penrose diagram one obtains the following set of mode functions in each quadrants:

$$u_{R,i}(k) = \begin{cases} e^{-i\omega t} f(k,r), & \text{in R} \\ 0, & \text{in L} \end{cases} \quad u_{L,i}(k) = \begin{cases} 0, & \text{in R} \\ e^{-i\omega t} f(k,r), & \text{in L} \end{cases}$$
$$u_{R,o}(k) = \begin{cases} e^{-i\omega t} f^*(k,r), & \text{in R} \\ 0, & \text{in L} \end{cases} \quad u_{L,o}(k) = \begin{cases} 0, & \text{in R} \\ e^{-i\omega t} f^*(k,r), & \text{in L} \end{cases} \tag{5.5}$$

Only two linear combinations can be built which meet the above criterium on holomorphicity. These are

$$\begin{cases} u_o = u_{R,o} + \alpha_o u_{L,o}, \\ u_i = u_{R,i} + \alpha_i u_{L,i}. \end{cases} \tag{5.6}$$

From the behaviour close to the horizon of the solutions and the analyticity requirement, the in-going and out-going cross-connecting functions α_i and α_o are constrained to be

$$\begin{cases} \alpha_o = e^{\frac{\pi\omega}{2}}, \\ \alpha_i = e^{-\frac{\pi\omega}{2}}. \end{cases} \tag{5.7}$$

In order to carry a similar analysis to the case of fermions, one must first check that solutions to the Dirac equation in an AdS-Schwarzschild background (5.1) behave as $e^{-i\omega r*}$ and its conjugate. In such a background[4] the spin coefficients are

$$\omega_{tr} = -r\left(1 + \frac{1}{r^4}\right) dt, \quad \omega_{ir} = r\sqrt{f} dx^i. \tag{5.8}$$

The Dirac equation then reads

$$\begin{cases} \left[-\frac{i\omega}{\sqrt{f}}\gamma^t + i\vec{\gamma}\cdot\vec{k}\right] A(m)\Psi_+(k,r) = \left[-\frac{\omega^2}{f} + \mathbf{k}^2\right]\Psi_-(k,r), \\ \left[-\frac{i\omega}{\sqrt{f}}\gamma^t + i\vec{\gamma}\cdot\vec{k}\right] A(-m)\Psi_-(k,r) = -\left[-\frac{\omega^2}{f} + \mathbf{k}^2\right]\Psi_+(k,r), \end{cases} \tag{5.9}$$

---

[4]Setting $R = 1$, $\pi T = 1$ for convenience.

where $A(m) = r\left[r\sqrt{f(r)}\partial_r + \frac{d-1}{2}\sqrt{f(r)} + \frac{(1+\frac{(\pi T)^4}{r^4})}{2\sqrt{f(r)}} - m\right]$. Focusing on the terms relevant for the near-horizon behaviour, solutions of the type $\frac{r}{\sqrt[4]{f(r)}}e^{\pm i\omega r^*}$ satisfy these equations. Note that it is crucial that the Γ^0 matrix be anti-hermitian. This leading near-horizon behaviour of solutions to the Dirac equation in a curved background is reminiscent of the forms of the solutions found in [33] in the course of this study of second-quantization for neutrino fields in a Kerr background. From the previous discussion on parallely-transported spinors in the complex U and V planes, one is led to consider the following set of mode functions in each quadrant:

$$\psi_{R,i} = \begin{cases} e^{-i\omega t}\sqrt{V}f(k,r)\imath, & \text{in R} \\ 0, & \text{in L} \end{cases} \qquad \psi_{L,i} = \begin{cases} 0, & \text{in R} \\ e^{-i\omega t}\sqrt{-V}f(k,r)\imath, & \text{in L} \end{cases}$$
$$\psi_{R,o} = \begin{cases} e^{-i\omega t}\sqrt{-U}f^*(k,r)\varsigma, & \text{in R} \\ 0, & \text{in L} \end{cases} \qquad \psi_{L,o} = \begin{cases} 0, & \text{in R} \\ e^{-i\omega t}\sqrt{U}f^*(k,r)\varsigma, & \text{in L} \end{cases} \tag{5.10}$$

The mergers $f(k,r)\left\{\begin{matrix}\sqrt{-U}\\\sqrt{V}\end{matrix}\right\}$ behave as $\frac{r}{\sqrt[4]{f(r)}}e^{\pi T t}e^{i\omega r^*}$, as required for solutions to the Dirac equation, except for the extra $e^{\pm \pi T t}$ term, which could be inserted in the definitions of the modes in (5.13) below. As for the scalar case reviewed above, the conditions that positive-frequency solutions are analytic in the lower U and V complex planes and negative-energy modes are analytic in their upper counterparts leads to the following linear combinations

$$\begin{cases} \psi_o = \psi_{R,o} + \beta_o\psi_{L,o}, \\ \psi_i = \psi_{R,i} + \beta_i\psi_{L,i}. \end{cases} \tag{5.11}$$

The behaviour of the solutions close to the horizon fixes

$$\begin{cases} \beta_o = ie^{\frac{\pi\omega}{2}}, \\ \beta_i = -ie^{-\frac{\pi\omega}{2}}. \end{cases} \tag{5.12}$$

The out-going and in-going solutions in (5.11) form a basis for a spinor field defined over the full Kruskal plane of the AdS-Schwarzschild geometry

$$\Psi_-(r) = \sum_k \left[a(\omega,\mathbf{k})\psi_o(k,r) + b(\omega,\mathbf{k})\psi_i(k,r)\right]. \tag{5.13}$$

In accordance with our discussion on the variational principle for spinor fields in AdS/CFT we do not expand the Ψ_+ field. Its leading-order part in an expansion near the boundary is fixed. One must fix the "position" and leave the "momentum" free to vary in a set of canonically conjugate pairs given by $\bar\chi_0$ and ψ_0. The coefficients $a(\omega,\mathbf{k})$, $b(\omega,\mathbf{k})$ are determined by requiring that (5.13) approaches $\Psi_-^R(k)$ and $\Psi_-^L(k)$ on their respective boundaries:

$$\begin{cases} \left[a(\omega,\mathbf{k})\sqrt{-U}\varsigma + b(\omega,\mathbf{k})\sqrt{V}\imath\right]_{r_{\partial M}} = \Psi_-^R(k), \\ -\left[a(\omega,\mathbf{k})e^{\pi\omega}\sqrt{U}(-)\varsigma - b(\omega,\mathbf{k})\sqrt{-V}(-)\imath\right]_{r_{\partial M}} = -ie^{-\frac{\pi\omega}{2}}\Psi_-^L(k), \end{cases} \tag{5.14}$$

The function $f(k, r)$ is normalized such that $f(k, r_{\partial M}) = 1$ at the boundary. The overall minus sign on the r.h.s. of the second equation results from the effect on spinor fields of time reversal from going to the R-quadrant to the L one:[5]

$$T^{-1}\Psi_{+,\alpha}(x)T = -\Psi_{+,\alpha}(\mathcal{T}x), \quad T^{-1}\Psi_{-}^{\dot{\alpha}}T = +\Psi_{-,\dot{\alpha}}(\mathcal{T}x). \qquad (5.15)$$

Also, recall that raising or lowering a spinor index comes with a minus sign. The meaning of (5.14) is as a set of two equations for two unknown spinors, $a(\omega, \mathbf{k})\sqrt{-U}\varsigma$ and $b(\omega, \mathbf{k})\sqrt{V}\imath$. The second equation in (5.14) involves the same unknown spinors but in the L quadrant, which introduces extra $\sqrt{-1}$. Taking care of those additional factors of i which occur in going from $\left\{ \begin{matrix} \sqrt{U} \\ \sqrt{-V} \end{matrix} \right\}$ to $\left\{ \begin{matrix} \sqrt{-U} \\ \sqrt{V} \end{matrix} \right\}$, (5.14) leads to

$$\begin{cases} a(\omega, \mathbf{k})\sqrt{-U} \mid_{r_{\partial M}} \varsigma = \frac{1}{e^{\pi\omega}+1}\left[\Psi_{-}^{R}(k) + e^{\frac{\pi\omega}{2}}\Psi_{-}^{L}(k) \right], \\ b(\omega, \mathbf{k})\sqrt{V} \mid_{r_{\partial M}} \imath = \frac{1}{e^{\pi\omega}+1}\left[e^{\pi\omega}\Psi_{-}^{R}(k) - e^{\frac{\pi\omega}{2}}\Psi_{-}^{L}(k) \right]. \end{cases} \qquad (5.16)$$

Here, $n(\omega) = \frac{1}{e^{\beta\omega}+1}$ is the Fermi-Dirac distribution. One may think that using (5.16) to replace , say, ς in the identification of l with $\frac{\partial}{\partial U}$ would lead to a constraint on the boundary data, which cannot be. Rather, the constraints are on the modes $a(\omega, \mathbf{k})$ and $b(\omega, \mathbf{k})$. The same happens for the scalar case first dealt with in [1], as is most apparent in equations (3.48) and (3.49) of [5] where the modes in an expansion of the scalar field in an in-going and out-going basis are related to the boundary values of the field in the right and left quadrants of a Kruskal diagram.

A computation of real-time Green functions from the standard AdS/CFT prescription is now in order. The classical boundary action in $r - t$ coordinates is

$$\begin{aligned} S_{\partial M} &= -i \int_{\partial M} \frac{d^d k}{(2\pi)^d} \sqrt{-g g^{rr}} \bar{\Psi}_{+}(-k, r) \Psi_{-}(k, r) \\ &= -i \int \frac{d^d k}{(2\pi)^d} \sqrt{-g g^{rr}} \bar{\Psi}_{+}(-k, r) \Psi_{-}(k, r) \mid_{r_R} -i \int \frac{d^d k}{(2\pi)^4} \sqrt{-g g^{rr}} \bar{\Psi}_{+}(-k, r) \Psi_{-}(k, r) \mid_{r_L} \end{aligned} \qquad (5.17)$$

For a scalar field the second integral would have come with the opposite sign. Here, however one must recall that spinor fields behave non-trivially when they cross the L quadrant where time ordering is reversed. While $\bar{\Psi}\Psi$ is invariant under time-reversal, what we are really considering instead is an expression where $\bar{\Psi}_{+}$ is fixed whereas Ψ_{-} is free to vary. The unusual sign is associated with the latter's transformation under time reversal, (5.15).

---

[5]Our conventions for the Clifford algebra differ from those of [34]. This affects in particular the Γ^5 matrix. Hence the overall sign flip in (5.15) as compared to the more familiar equation (40.32) from [34].

Using (5.13) and (5.16) the boundary action becomes

$$S_{\partial M} = -i\int \frac{d^4k}{(2\pi)^4}\sqrt{-g g^{rr}}\,n(\omega)u_{R,o}(k)\left[\bar\Psi_+^R(-k)\Psi_-^R(k) + e^{\frac{\beta\pi}{2}}\bar\Psi_+^R(-k)\Psi_-^L(k)\right] \qquad (5.18)$$

$$-i\int \frac{d^4k}{(2\pi)^4}\sqrt{-g g^{rr}}\,n(\omega)u_{R,i}(k)\left[e^{\frac{\beta\omega}{2}}\bar\Psi_+^R(-k)\Psi_-^R(k) - e^{\beta\omega}\bar\Psi_+^R(-k)\Psi_-^L(k)\right]$$

$$+i\int \frac{d^4k}{(2\pi)^4}\sqrt{-g g^{rr}}\,n(\omega)u_{L,o}(k)e^{\frac{\beta\omega}{2}}\left[\bar\Psi_+^L(-k)\Psi_-^R(k) + e^{\frac{\beta\omega}{2}}\bar\Psi_+^L(-k)\Psi_-^L(k)\right]$$

$$-i\int \frac{d^4k}{(2\pi)^4}\sqrt{-g g^{rr}}\,n(\omega)u_{L,i}(k)e^{-\frac{\beta\omega}{2}}\left[e^{\beta\omega}\bar\Psi_+^L(-k)\Psi_-^R(k) - e^{\frac{\beta\omega}{2}}\bar\Psi_+^L(-k)\Psi_-^L(k)\right].$$

Equations (3.26) and (2.8), which for fermionic operators reads $G_{A,ab}(k) = G^*_{R,ba} = -iS^*(k)\gamma^t_{ba} = iS^*(k)\gamma^t_{ab}$ — from Γ^0 being anti-hermitian, cf., e.g., (3.11) — and the near-boundary expansions of $\Psi_\pm^{R,L}$ yield

$$\begin{cases} G_{RR}(k) = -i\left[n(\omega)e^{\beta\omega}(-i)G_R(k) + n(\omega)(-i)G_A(k)\right] = -\text{Re}\,G_R(k) + i\tanh(\frac{\omega}{2T})\text{Im}\,G_R(k), \\ G_{LL}(k) = -i\left[-n(\omega)(-i)G_R(k) - n(\omega)e^{\beta\omega}(-i)G_A(k)\right] = \text{Re}\,G_R(k) + i\tanh(\frac{\omega}{2T})\text{Im}\,G_R(k), \\ G_{RL}(k) = -in(\omega)e^{\frac{\beta\omega}{2}}\left[(-i)G_R(k) - (-i)G_A(k)\right] = -\frac{2ie^{\frac{\beta\omega}{2}}}{1+e^{\beta\omega}}\text{Im}\,G_R(k), \\ G_{LR}(k) = -in(\omega)e^{\frac{\beta\omega}{2}}\left[-(-i)G_R(k) + (-i)G_A(k)\right] = \frac{2ie^{\frac{\beta\omega}{2}}}{1+e^{\beta\omega}}\text{Im}\,G_R(k). \end{cases} \qquad (5.19)$$

These are the Schwinger-Keldysh propagators (2.9) with $\sigma = \frac{\beta}{2}$ for a fermionic operator dual to the supergravity field Ψ. One can redefine $\Psi^L(k)$ to obtain real-time propagators with arbitrary $0 < \sigma < \beta$. Let us illustrate how (5.19) is obtained and focus on $G_{RR}(k)$. The relevant terms from the boundary action are

$-in(\omega)\sqrt{-g g^{rr}}\left[u_{R,i}(k)e^{\beta\omega}\bar\Psi_+^R(-k)\Psi_-^R(k) + u_{R,o}(k)\bar\Psi_+^R(-k)\Psi_-^R(k)\right].$

The first one comes with an in-going mode. One then uses the near-boundary expansion for $\Psi_\pm^R$ and the relation (3.20), i.e. $\psi_0(k) = \mathcal{S}(k)\chi_0(k)$, to translate this term into an expression proportional to the retarded propagator. On the other hand, the second term in brackets involves an out-going mode. It must be associated with an advanced Green function [35]. One then writes $\sqrt{-g g^{rr}}\bar\Psi_+^R(-k)\Psi_-^R(k) = \sqrt{-g g^{rr}}\bar\Psi_-^R(k)\Psi_+^R(-k)$ which is equal to $\mathcal{S}^*(k)\bar\chi_0^R(k)\chi_0^R(-k) = \mathcal{S}^*(k)\bar\chi_0^R(-k)\chi_0^R(k)$. This finally leads to the stated result. We have thus checked that Schwinger-Keldysh correlators for fermionic operators and the standard relations among them and the retarded, advanced and symmetric propagators hold in AdS/CFT by taking functional derivatives of boundary part of the dual supergravity action. This prescription can be generalized to higher-point functions of a fermionic operator, provided the non-quadratic parts of the action for its dual supergravity spinor field are known.

Acknowledgments

– 14 –

References

[1] C.P. Herzog and D.T. Son, *Schwinger-Keldysh propagators from AdS/CFT correspondence*, *JHEP* **03** (2003) 046 [hep-th/0212072] [SPIRES].

[2] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, hep-th/0201253 [SPIRES].

[3] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled $N = 4$ Yang-Mills*, *Phys. Rev.* **D 74** (2006) 085012 [hep-ph/0605199] [SPIRES].

[4] S.S. Gubser, *Momentum fluctuations of heavy quarks in the gauge-string duality*, *Nucl. Phys.* **B 790** (2008) 175 [hep-th/0612143] [SPIRES].

[5] J. Casalderrey-Solana and D. Teaney, *Transverse momentum broadening of a fast quark in a $N = 4$ Yang-Mills plasma*, *JHEP* **04** (2007) 039 [hep-th/0701123] [SPIRES].

[6] J. de Boer, V.E. Hubeny, M. Rangamani and M. Shigemori, *Brownian motion in AdS/CFT*, *JHEP* **07** (2009) 094 [arXiv:0812.5112] [SPIRES].

[7] D.T. Son and D. Teaney, *Thermal noise and stochastic strings in AdS/CFT*, *JHEP* **07** (2009) 021 [arXiv:0901.2338] [SPIRES].

[8] G.C. Giecold, E. Iancu and A.H. Mueller, *Stochastic trailing string and Langevin dynamics from AdS/CFT*, *JHEP* **07** (2009) 033 [arXiv:0903.1840] [SPIRES].

[9] P.M. Chesler and L.G. Yaffe, *Horizon formation and far-from-equilibrium isotropization in supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **102** (2009) 211601 [arXiv:0812.2053] [SPIRES].

[10] G.C. Giecold, *Heavy quark in an expanding plasma in AdS/CFT*, *JHEP* **06** (2009) 002 [arXiv:0904.1874] [SPIRES].

[11] A. Akhavan, M. Alishahiha, A. Davody and A. Vahedi, *Fermions in non-relativistic AdS/CFT correspondence*, arXiv:0902.0276 [SPIRES].

[12] M. Cubrovic, J. Zaanen and K. Schalm, *Fermions and the AdS/CFT correspondence: quantum phase transitions and the emergent Fermi-liquid*, arXiv:0904.1993 [SPIRES].

[13] R.G. Leigh and N.N. Hoang, *Real-time correlators and non-relativistic holography*, arXiv:0904.4270 [SPIRES].

[14] K. Skenderis and B.C. van Rees, *Real-time gauge/gravity duality*, *Phys. Rev. Lett.* **101** (2008) 081601 [arXiv:0805.0150] [SPIRES].

[15] K. Skenderis and B.C. van Rees, *Real-time gauge/gravity duality: prescription, Renormalization and Examples*, *JHEP* **05** (2009) 085 [arXiv:0812.2909] [SPIRES].

[16] B.C. van Rees, *Real-time gauge/gravity duality and ingoing boundary conditions*, *Nucl. Phys. Proc. Suppl.* **192-193** (2009) 193 [arXiv:0902.4010] [SPIRES].

[17] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [arXiv:0804.3972] [SPIRES].

[18] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099] [SPIRES].

[19] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non- relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100] [SPIRES].

[20] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111] [SPIRES].

[21] M. Le Bellac, *Thermal field theory*, Cambridge University Press, Cambridge U.K. (1996).

[22] N.P. Landsman and C.G. van Weert, *Real and imaginary time field theory at finite temperature and density*, *Phys. Rept.* **145** (1987) 141 [SPIRES].

[23] L.P. Kadanoff and G. Baym, *Quantum statistical mechanics: Green's function methods in equilibrium and nonequilibrium problems*, W.A. Benjamin, Inc. (1962).

[24] M. Henningson and K. Sfetsos, *Spinors and the AdS/CFT correspondence*, *Phys. Lett.* **B 431** (1998) 63 [hep-th/9803251] [SPIRES].

[25] M. Henneaux, *Boundary terms in the AdS/CFT correspondence for spinor fields*, hep-th/9902137 [SPIRES].

[26] N. Iqbal and H. Liu, *Real-time response in AdS/CFT with application to spinors*, *Fortsch. Phys.* **57** (2009) 367 [arXiv:0903.2596] [SPIRES].

[27] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, *Phys. Rev.* **D 79** (2009) 025023 [arXiv:0809.3808] [SPIRES].

[28] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Cie (1973) [SPIRES].

[29] J. Polchinski, *String theory, Volume 2: Superstring theory and beyond*, Cambridge University Press, Cambridge U.K. (2004) [SPIRES].

[30] R. Penrose and W. Rindler, *Spinors and space-time: Volume 2, Spinor and twistor methods in space-time geometry*, Cambridge University Press, Cambridge U.K. (1988) [SPIRES].

[31] R. Penrose and W. Rindler, *Spinors and space-time: Volume 1, Two-spinor calculus and relativistic fields*, Cambridge University Press, Cambridge U.K. (1988) [SPIRES].

[32] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566 [SPIRES].

[33] W.G. Unruh, *Second quantization in the Kerr metric*, *Phys. Rev.* **D 10** (1974) 3194 [SPIRES].

[34] M. Srednicki, *Quantum field theory*, Cambridge University Press, Cambridge U.K. (2007) [SPIRES].

[35] D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence: recipe and applications*, *JHEP* **09** (2002) 042 [hep-th/0205051] [SPIRES].